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Applications of Partial Differential Equations in Reservoir Simulation

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Summary

The solution to stochastic partial differential equations may be viewed in several manners. One can view a solution as a random field (set of random variables indexed by a multidimensional parameter). In the case where the SPDE is an evolution equation, the infinite dimensional point of view consists in viewing the solution at a given time as a random element in a function space and thus views the SPDE as a stochastic evolution equation in an infinite dimensional space. In the path wise point of view, tries to give a meaning to the solution for (almost) every realization of the noise and then view the solution as a random variable on the set of (infinite dimensional) paths thus defined.

The main focus of this paper is on mathematical formulation of the physical model is obtained, usually involving coupled systems of non-linear partial differential equations. This can be best done by using Stochastic Partial Differential Equations, which can take care of randomness of the event and gives the most precise measure of the results.

Introduction

A stochastic partial differential equation (SPDE) is a partial differential equation containing a random (noise) term. The study of SPDEs is an exciting topic which brings together techniques from probability theory, functional analysis, and the theory of partial differential equations.

Over the past two decades the exploration of stochastic partial differential equations (SPDEs) has become one of the most rapidly expanding areas in Probability Theory. In addition to applications to numerous problems in Mathematical Physics and Life Sciences, interest in such studies is motivated by a desire to understand and control the behavior of complex systems that appear in many areas of natural and social sciences.

The objective of reservoir simulation is to understand the complex chemical, physical and fluid flow processes occurring in a petroleum reservoir sufficiently well to be able to optimize the recovery of hydrocarbon. In order to predict the reservoir performance, a series of models of reservoir processes are constructed which yield information about the complex phenomena accompanying different recovery methods. In this paper, various approaches to the use of mathematical models for reservoir simulation problems are presented and compared. There are four major stages to the modeling process for reservoir simulation. First, a physical model of the flow processes is developed incorporating as much physics is deemed necessary to describe the essential phenomena. Second, a mathematical formulation of the physical model is obtained, usually involving coupled systems of non-linear partial differential equations. Third, once the properties of mathematical model, such as existence, uniqueness and regularity of the solution are sufficiently well understood and the properties seem compatible with the physical model, a discretized numerical model of the mathematical equations is produced. Finally, a computer program capable of efficiently performing the necessary computations for the numerical model is sought.

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Theory

The modeling process is not complete just passing the four described above steps. Once a computer program has been developed which gives concrete quantitative results for the total model, this output should be compared with measured observations of the physical process. If the results do not compare extremely well, one should iterate back through the complete modeling process, changing the various intermediate models in ways to obtain better correlation between physical measurements and the computational results. Usually many iterations of this modeling loop are necessary to obtain reasonable models for the highly complex physical phenomena.

3.1 Description of problems in reservoir simulation

a) In general, the hydrocarbon is trapped in the microscopic pores of a rock (which further defines the porosity of the rocks) and will flow through the rock only under the influence of extremely large pressure differentials. Also, as the geology in the reservoir changes, the reservoir often has areas of high flow and areas of reduced flow. This heterogeneous nature of most reservoirs greatly complicates the mathematical modeling process, especially since the reservoir is inaccessible and first -hand information about its porosity can only be obtained through the wells, which may be hundreds of yards apart.

b) In water flooding, if flow rate is sufficiently high, the interface between the resident petroleum and the invading water becomes unstable and tends to form long fingers which grow in length towards the production wells, bypassing much of the hydrocarbons. The production of petroleum from that well is then greatly reduced. This phenomenon is termed as *viscous fingering* and is serious problem in hydrocarbon recovery.

c) Among the various types of recovery techniques available involve many different physical processes; there are certain common mathematical problems which must be addressed. First, the EOR (Enhanced Oil Recovery) displacement processes are dominated by convective flow from the injection wells to the production wells. Therefore, mathematical models for each of these enhanced recovery techniques must have strong transport terms and must possess many of the properties of first order hyperbolic partial differential equations. Another important common phenomenon in each of the models is the importance of the location of the interface between the injected fluid and the resident fluid. The location of this interface indicates how much of and where the hydrocarbons are left in the reservoir as a function of time.

3.2 Development of Representative Model Equations

Fluid motions in porous media are governed by the same fundamental laws that govern their flow in pipelines and rivers. These laws are based on conservation of mass, momentum and energy. Additional governing equations are rate equations, generally a form of Darcy's Law and equations of state. Since we usually do not have a complete knowledge of the total behavior of the system, a major difficulty in the modeling procedures is the choice of a set of equations which accurately describes the complex physical process. From Darcy's Law superficial fluid velocity of a homogenous fluid with volumetric flow Q passing normally through a cross sectional area A is given by-

$$\vec{u} = \frac{Q}{A} = -\frac{k}{\mu}(\nabla p - \rho g \nabla Z)$$

…(3.2.1)

Where $\boldsymbol{\mu}$ is the viscosity of the fluid, p is the fluid pressure, $\boldsymbol{\rho}$ is the fluid density, g is the magnitude of the acceleration due to gravity, the depth **Z** is a vector function of (x, y, z) pointing in the direction of gravity, and $\boldsymbol{k}$ is a absolute permeability tensor, which in most uses is assumed that $\boldsymbol{k}$ is the special diagonal tensor

$$k = \begin{bmatrix} k_x & 0 & 0 \\ 0 & k_y & 0 \\ 0 & 0 & k_z \end{bmatrix}$$

…(3.2.2)

Where $\boldsymbol{k}$x, $\boldsymbol{k}$y, and $\boldsymbol{kz}$ are interpreted as permeabilities in the x, y and z direction respectively.

If φ is the porosity of the material denotes the fraction of volume $\boldsymbol{V}$ available for flow and $\boldsymbol{q}$ is the mass flow rate per unit volume injected (or produced from) $\boldsymbol{V}$, then from equation of conservation of mass we have-

$$\frac{d}{dt}\int_V \phi\rho(x,t)dx = -\int_{\partial V} \rho(x,t)\tilde{u}\cdot\nu ds + \int_V q dx.$$

…(3.3.3)

If we use divergence theorem to see that

$$\int_{\partial V} \rho\tilde{u}\cdot\nu ds = \int_V \nabla\cdot(\rho\tilde{u})dx,$$

…(3.3.4)

And interchange d/dt with the spatial integration,

$$\frac{d}{dt}\int_V \phi\rho(x,t)dx = \int_V \frac{\partial(\phi\rho)}{\partial t}dx,$$

…(3.3.5)

We obtain

$$\int_V \frac{\partial(\phi\rho)}{\partial t}dx = \int_V \left[-\nabla\cdot(\rho\tilde{u}) + q\right]dx.$$

…(3.3.6)

Since equation (3.3.6) is to hold for any volume element V, we obtain the partial differential equation (pde) form for single – phase flow subject to a superficial velocity $\tilde{u}$

$$\frac{\partial(\phi\rho)}{\partial t} = -\nabla\cdot(\rho\tilde{u}) + q$$

…(3.3.7)

Combining (3.3.1) and (3.3.7), we obtain

$$\frac{\partial(\phi\rho)}{\partial t} = \nabla\cdot\frac{\rho k}{\mu}(\nabla p - \rho g\nabla Z) + q, \qquad x\in\Omega, \quad t\in[t_0,t_1],$$

…(3.3.8)

Where Ω is our spatial domain and $[t_0, t_1]$ is the time interval under consideration.

Note that (3.3.8) is a second order pde in the two dependent variables p = pressure and ρ = density. By expressing p as a function of ρ, or vice versa, we can obtain a pde in only one independent variable. This is accomplished by use of an equation of state describing the relationship between p and ρ.

As an equation of state, we shall use the definition of fluid compressibility, c, at a fixed temperature T:

$$c = -\frac{1}{V}\frac{\partial V}{\partial p}\bigg|_T$$

…(3.3.9)

This can be expressed in terms of density:

$$c = \frac{1}{\rho}\frac{\partial \rho}{\partial p}\bigg|_T$$

…(3.3.10)

Separating variables in (3.3.10) and denoting ρ_0 as the density at pressure p_0, we obtain

$$p = p_0 + \frac{1}{c}\ln\left(\frac{\rho}{\rho_0}\right)$$

…(3.3.11)

Or equivalently,

$$\rho = \rho_0 \exp\left(c(p - p_0)\right).$$

…(3.3.12)

Thus there is a one-to-one, invertible mapping between p and ρ described by (3.3.11) or (3.3.12). Therefore using the chain rule, (3.3.8) can be written as single pde for the density

$$\frac{\partial(\phi\rho)}{\partial t} = \nabla\cdot\frac{k}{\mu c}(\nabla p - \rho^2 cg\nabla Z) + q, \qquad x\in\Omega, \quad t\in[t_0,t_1],$$

…(3.3.13)

Since (3.3.13) is a parabolic pde it requires an initial specification of the density throughout the domain and boundary conditions.

$$\rho(x,0) = \rho_0(x), \qquad x\in\Omega,$$

…(3.3.14)

Usually, Nuemann boundary conditions or the specifications of the mass flow across the boundary, is used:

$$\rho \tilde{u} \cdot \nu|_{\partial\Omega} = g_1(x,t), \qquad x \in \partial\Omega, \quad t \in [t_0, t_1].$$

…(3.3.15)

However, in some situations when the density (or pressure) can be measured at the boundary, Dirichlet conditions are imposed. If the compressibility of the fluid is small as it is for man y liquids, then the Taylor series expansion for the exponential in (3.3.12) ca n be truncated and the assumption can be made that

$$\rho = \rho_0(1 + c(p - p_0)).$$

…(3.3.16)

Under this assumption, a pde for pressure can be written from (3.3.13)

$$c\frac{\partial(\phi\rho)}{\partial t} - \nabla \cdot \frac{k}{\mu}(\nabla p - \rho_0(1 + c(p - p_0))g\nabla Z) + c\nabla p \cdot (\nabla p - \rho_0(1 + c(p - p_0))g\nabla Z) + \frac{q}{\rho}, \quad x \in \Omega, t \in [t_0, t_1].$$

…(3.3.17)

Under the even stricter assumption that the fluid is incompressible (c=0), we can obtain from (3.3.17) the elliptic pde

$$0 = \nabla \cdot \frac{k}{\mu}(\nabla p - p_0 g \nabla Z) + \frac{q}{\rho}, \qquad x \in \Omega.$$

…(3.3.18)

This is also the form of the equation for *steady-state* $(\partial(\varphi\rho)/\partial t = 0)$ flow. Thus for either incompressible fluids or steady-state flow, from (3.3.3) we see that if Neumann boundary conditions or mass flow rates are specified on all of the boundary, then for a solution to equation (3.3.18) to exist, we must have

$$\int_V q(x,t)dx = 0, \qquad t \in [t_0, t_1],$$

And flow into all of t he injection wells must be exactly balanced by flow out of all of the production wells. Also for (3.3.18) with only flow boundary conditions, the pressure is only determined to within a constant; if $\tilde{p}$ satisfies (3.3.18), then $\tilde{p} + \tilde{c}$ satisfies (3.3.18) for any constant $\tilde{c}$. This lack of uniqueness causes no problems as long as it is understood and taken in to account in the simulation.

If we consider (3.3.6) as the governing equation instead of its counterpart (3.3.7) the "point source" interpretation is not so crucial as long as the well is not at the boundary of the computational element. Most of the simulation done in the petroleum industry today uses a different type of well model which can more easily accommodate the specification or computation of bottom hole pressures at the wells. These models are based on analytical solutions for steady-state radial flow equations and incorporate the size of the computational grid blocks and the true well bore radius. The simplest versions of these models relate the pressure p at a distance r from the well to the volumetric flow rate q via the equation

$$p_e - p = \frac{q\mu}{2\pi hk} \ln\left(\frac{r_e}{r}\right),$$

…(3.3.19)

Where p_e is the pressure at the "effective" drainage radius r_e and h is the thickness of the region of assumed radial flow. The single phase flow equations developed above are not sufficient to model the simultaneous interactive flow of two or more phases; this type of flow dominates essentially all the Enhanced Recovery Processes (ERO). So, we need to derive equations for multi- phase flow in porous media.

Let us first assume that we have two immiscible fluid phases flowing simultaneously and there is no mass transfer between the fluids. Fluid wetting the porous medium more than the other is termed as wetting phase fluid (subscript used w) and the other is known as non- wetting fluid (subscript used n). The saturation of a phase is the fraction of the space available to flow occupied by that phase. Since both phases are flowing we have,

$$S_w + S_n = 1.$$

…(3.3.20)

We also have separate pressures for each phase p_w and p_n. The difference between these pressures is the capillary pressure p_c.

$$p_c(S_w) = p_n - p_w.$$

…(3.3.21)

Each phase will have its density, viscosity and Darcy's velocity analogous of (3.3.1), which can be written as:

$$\tilde{u}_w = -\frac{k_w}{\mu_w}(\nabla p_w - \rho_w g \nabla Z),$$

…(3.3.22a)

$$\tilde{u}_n = -\frac{k_n}{\mu_n}(\nabla p_n - \rho_n g \nabla Z),$$

…(3.3.22b)

Where symbols have their usual scientific meanings. In the simultaneous flow of two fluids, each flow interferes with the flow of the other and the effective permeabilities are less than or equal to the single phase permeability k of porous medium. We define relative permeabilities as

$$k_{rw} = \frac{k_w}{k} \leq 1,$$

…(3.3.23a)

$$k_{rn} = \frac{k_n}{k} \leq 1,$$

…(3.3.23b)

The same arguments which led to (3.3.8) will t hen yield the system, for x $\in$ Ω and t $\in$ [t0, t1],

$$\frac{\partial(\phi\rho_w S_w)}{\partial t} = \nabla \cdot \frac{\rho_w k k_{rw}}{\mu_w}(\nabla p_w - \rho_w g \nabla Z) + q_w,$$

$$\frac{\partial(\phi\rho_n S_n)}{\partial t} = \nabla \cdot \frac{\rho_n k k_{rn}}{\mu_n}(\nabla p_n - \rho_n g \nabla Z) + q_n$$

…(3.3.24a&b)

The equations are coupled via the constraints (3.3.20) and (3.3.21) and the pressures and densities for each phase are related by equations of state of the form

$$c_w = \frac{1}{\rho_w}\frac{\partial \rho_w}{\partial p_w}\bigg|_T,$$

$$c_n = \frac{1}{\rho_n}\frac{\partial \rho_n}{\partial p_n}\bigg|_T.$$

…(3.3.25a&b)

Since the system (3.3.24) resembles equation (3.3.11) closely one might think that it is a system of parabolic pde's with diffusion- like properties. This is not necessarily the case. We can see the properties more clearly under the assumptions that the densities of the phases and the porosity are constants and there are no gravity terms. Then defining the total and phase mobilities as

$$\lambda(S_w) = \frac{k_{rn}(S_w)}{\mu_n} + \frac{k_{rw}(S_w)}{\mu_w},$$

$$\lambda_i(S_w) = \frac{k_{ri}(S_w)}{\lambda(S_w)\mu_i}, \qquad i = n, w,$$

…(3.3.26a&b)

Defining the average pressure as

$$\bar{p} = \frac{1}{2}(p_w + p_n) + \frac{1}{2}\int_0^{p_c}(\lambda_n(\xi) - \lambda_w(\xi))\,d\xi,$$

…(3.3.27)

And defining the total fluid velocities as

$$\tilde{u} = -k(x)\lambda(S_w)\nabla p,$$

…(3.3.28)

We can add and subtract the equations in (3.3.24) and collect terms to obtain for x $\in$ Ω and t $\in$ [t0, t1],

$$\nabla \cdot \tilde{u} = -\nabla \cdot (k(x)\lambda(S_w)\nabla p) = \frac{q_n}{\rho_n} + \frac{q_w}{\rho_w},$$

$$\phi\frac{\partial S_w}{\partial t} - \nabla \cdot (k\lambda\lambda_n\lambda_w\frac{\partial p_c}{\partial S_w}\nabla S_w) + \lambda_w\tilde{u} \cdot \nabla S_w = f(q_w, p_w, \lambda_w, q_n, p_n\lambda_n),$$

…(3.3.29a&b)

Where f is a simple linear function of the flow properties. If either fluid or the rock is compressible then (3.3.29a) becomes:

$$\nabla \cdot k\lambda \nabla p - \left(\frac{q_n}{\rho_n} + \frac{q_w}{\rho_w} \right) + \phi c_t \frac{\partial p}{\partial t}$$

…(3.3.30)

Where c_t is the total compressibility of the system given as

$$c_t = \frac{1}{\phi} \frac{\partial \phi}{\partial p} + (S_n c_n + S_w c_w).$$

…(3.3.31)

The equation for saturation (3.3.29b) is similar to a non linear convection- diffusion equation. The governing equations for the miscible displacement of one incompressible fluid by another in a porous medium are given by:

$$\nabla \cdot \tilde{u} = -\nabla \cdot \frac{k(x)}{\mu(S)} \nabla p = q,$$

$$\phi \frac{\partial S}{\partial t} - \nabla \cdot (D \nabla S) + \tilde{u} \cdot \nabla S = (\tilde{S} - S) q,$$

…(3.3.32a&b)

For $x \in \Omega$ and $t \in [t_0, t_1]$, where p and $\tilde{u}$ are the total fluid pressure and velocity, S is the saturation of the invading fluid, $\tilde{S}$ is the specified injection or resident production saturation of the injected fluid and q is the total volumetric flow rate at the well. D is the diffusion- dispersion tensor given by:

$$(D_{ij}(x, \tilde{u}) = \phi d_m I + \frac{d_l}{|\tilde{u}|} \begin{pmatrix} u_1^2 & u_1 u_2 \\ u_1 u_2 & u_2^2 \end{pmatrix} + \frac{d_t}{|\tilde{u}|} \begin{pmatrix} u_2^2 & -u_1 u_2 \\ -u_1 u_2 & u_1^2 \end{pmatrix},$$

…(3.3.33)

Where $\tilde{u} = (u_1, u_2)$, $|\tilde{u}|$ is the Euclidean norm of $\tilde{u}$, d_m is the molecular diffusion coefficient and d_1 and d_2 are the magnitudes of longitudinal and transverse dispersion.

As before, if the fluids are compressible, the elliptic pde (3.3.32) will become parabolic. In general is d_m is assurned to be very small with d_1 and d_2 somewhat larger. Since the magnitudes of the last two terms in D are approximately $d_1 |\tilde{u}|$ or $d_2 |\tilde{u}|$ we see more dispersive mixing where the velocities are higher around the wells and less out in the reservoir. Therefore although the systems (3.3.29) and (3.3.32) appear to be very similar, there are some important differences. Both are transport dominated but not purely hyperbolic systems.

4. Conclusions

So, the mathematical modeling for Reservoir Simulation is explained in the above discussion. But even after taking so much care and possibilities some problems still prevails during the simulation and hence on the results and interpretations. Some of them are as follows:

a) The artificial diffusion or numerical diffusion does stabilize the difference method but the numerical error induced can be disastrous to reservoir simulation and can destroy the physical information in the model (Fig.1).

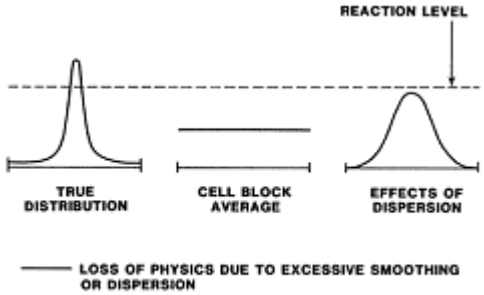

Fig.1 Front Resolution – Thermal simulation

b) Another disastrous effect of upstream weighting of the transport term is a phenomenon of obtaining drastically different numerical predictions from simulators due only to different spatial orientations of the computational grid, if upstream weighting is used for transport terms in each of the x, *y*. and *z* directions, and one will obtain an artificial dispersion term of the form

$$d = -\frac{\Delta x}{2} \frac{\partial^2 S}{\partial x^2} - \frac{\Delta y}{2} \frac{\partial^2 S}{\partial y^2} - \frac{\Delta z}{2} \frac{\partial^2 S}{\partial z^2}$$

Which is not rotationally invariant and is thus directionally dependent. (See Fig.2)

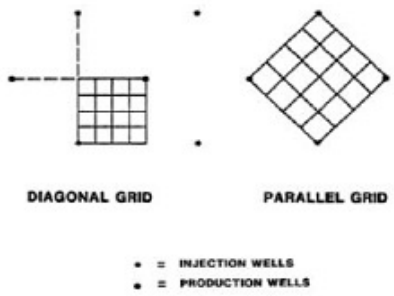

Fig.2 Grid orientation effect – uniform five-spot pattern.

A variety of methods sequentially solving the governing equations in an implicit manner without the full coupling have also been developed. These methods have less stability but better computational features than fully coupled, fully implicit methods but more stability and somewhat worse computational properties than Implicit Pressure Explicit Saturation (IMPES).

References

Albright, N., Anderson C., and Conus P.,*The random method for calculating fluid displacement in a porous medium in Boundary and Interior Layers – Computational and Asymptotic methods*, J.J.H. Miller,ed., Boole Press, Dublin,1980, pp. 3-13.

Aziz, K., and Shettari, A., *Petroleum Reservoir Simulation*, Applied Science Publishers, London, 1979.

Chavent, G., *A new formulation of diphasic incompressible flows in porous media*, Lecture notes in Mathematics503, Springer- Verlag, Berlin, 1976, pp. 258-270.

Darlow, B.L., *A penalty- Galerkin method for solving the miscible displacement problem*, Ph.D. thesis, Rice University, Houston,
1980.

Douglas, J., Jr., Darlow, B.L., Wheeler, M.F., and Kendall, R.P., *Self-adaptive Galerkin methods for one dimensional two phase immiscible flow,* SPE 7679, 5th SPE Symposium on Reservoir Simulation, Denver, 1979.

Douglas, J., Jr., Darlow, B.L., and Rachford, H.H., Jr.,The *application of variational methods to water flooding problems*, J.Canad. Pet.Tech., 8 (1969), pp 79-85.

Ewing, R.E. and Russel, T.F., *Determination of coefficients in reservoir simulation in Numerical Treatment of Inverse Problems in porous media*, SIAM J. Numer. Anal., 19 (1982), pp.1-66.

Glimm, J., Marchesein, D., and McBryan, O., *Statistical fluid dynamics: unstable fingers,* Comm. Math. Phys.,74 (1980),pp.1-13.

Nolan, J.S., Numerical simulation of compositional phenomena in petroleum reservoirs, SPE 4274, 3nd Symposium on Numerical Simulation of Reservoir Performance, Houston, 1973.

Perkins, T.K., and Johnson, O.C., A review of diffusion and dispersion in porous media, SPE, 3 (1963), pp. 70-84.