9th Biennial International Conference
& Exposition on Petroleum Geophysics

"HYDERABAD 2012"

P-127

Exponential Finite Difference Method for Simulation of Electromagnetic Response of Layered Earth

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Summary

This paper presents an efficient algorithm Exponential Finite Difference Method (EFDM) for simulation of electromagnetic response of layered earth by considering the exponential basis function. EFDM demands a parameter μ to be chosen judiciously to obtain optimum results. The estimators of optimum μ can be obtained from eigenvalue analysis of the coefficient matrix; however, near optimal values can be constructed using model parameters. Since the electromagnetic response has oscillatory behavior, EFDM handles it better and gives more accurate results in comparison to the Classical Finite Difference Method (CFDM). Using EFDM we can choose coarser grids to obtain same accuracy of result as CFDM provides with a given grid. As a result EFDM reduces the time and cost of computation in comparison to CFDM.

Key words: *Partial Differential Equations, Finite Difference Method, exponential fitting, electromagnetic response*

Introduction

In geophysical problems usually we encounter partial differential equations (PDEs). For a multilayered earth or a multidimensional earth obtaining analytical solution is very difficult or not possible at all. In such situations numerical methods are applied to get a solution. In general, these numerical methods transform the governing differential equation into a matrix equation which is then solved to obtain the solution. The quality of solution depends on the numerical methods employed for transformation and efficiency of the method is judged by the factors like time of computation, accuracy of solution, memory of computer used. The first two factors are always coupled in the sense that to get very accurate solution, computation time will be enormous; and to get in minimum time, an appreciable compromise with accuracy has to be made. The numerical scheme that optimizes both the factor is the best one to use.

Classical Finite Difference Method (CFDM) is one of the numericalmethods for solving partial differential equations. It assumes that field to be solved does not have oscillating character and behaves as low degree polynomial. However, when field has oscillating or hyperbolic behavior (as in the case of Electromagnetic or Seismic) then accuracy of the solution obtained using CFDM is not very good. It will give good result only when domain of interest is very finely discretized so that within each cell field can be assumed as low degree polynomial. Thus it leads to a coefficient matrix of very large size and will take quite an amount of time to solve. For 2D/3D geophysical problems this situation will be worse and require enormous amount of time to solve.

In this paper we propose the use of a new method Exponential Finite Difference Method (EFDM) based on the work of Ixaru, 1997 and Ixaru and Berghe, 2004 where they approximated the function by considering the exponential basis function. EFDM can handle oscillating nature of field and we can discretize domain of interest with coarser grid to get same accuracy of result as with the CFDM in sufficiently less time.

EFDM requires a parameter μ that need to be chosen properly in accordance with the characteristics of field. Estimators of optimum μ are constructed using model parameters as well as eigenvalue analysis of coefficient matrix so that it will give optimum relative error.

We performed experiments on numerous random models of layered earth for solving 1D Helmholtz equation using both CFDM as well as proposed EFDM to observe the efficiency of proposed method.

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Theory

To test proposed EFDM we consider 1D Helmholtz equation. For ith layer field has following equation,

$$\frac{d^2u_i(x)}{d^2x} + k_i^2 u_i(x) = 0 \qquad (1)$$

Where $u_i(x)$ is field value we seek to obtain and,
$k_i = (1+i)/\delta_i$
$\delta_i \approx 503\sqrt{\rho_i T}$
ρ_i= i^{th} layer resistivity and
T= time period

The first step of solving PDEs is discretization of domain of interest as shown in figure 1. Then the field is computed at all nodes (shown in figure 1 as red points).

Solving PDE using Classical Finite Difference Method (CFDM)

The central difference formula for second derivative is

$$u''(x_i) = \frac{1}{H_i^2}\left[\frac{h_{i+1}}{n_i}u(x_{i-1}) - u(x_i) + \frac{h_i}{n_i}u(x_{i+1})\right], \qquad (2)$$

where

$h_i = (x_i - x_{i-1})$, $H_i^2 = \frac{1}{2}(h_{i+1}.h_i)$ and $n_i = (h_i + h_{i+1})$

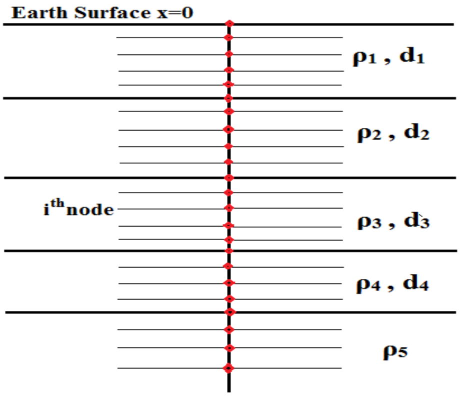

Figure 1: Layered earth model and grid generation. Thin lines show nodal line, thick lines show layer interface and red points show node points.

Using equation 1 and equation 2

$$\frac{1}{H_i^2}\left[\frac{h_{i+1}}{n_i}u(x_{i-1}) - u(x_i) + \frac{h_i}{n_i}u(x_{i+1})\right] + k_i^2 u(x_i) = 0$$

$$\frac{h_{i+1}}{n_i}u_{i-1} + (k_i^2 H_i^2 - 1)^2 u_i + \frac{h_i}{n_i}u_{i+1} = 0 \qquad (3)$$

In equation 3, $I = 1, 2, \ldots., n$ and, u_0 and u_{n+1} act as boundary conditions which need be supplied to solve system of equations,

$$\mathbf{M}.\mathbf{u} = \mathbf{b}, \qquad (4)$$

where **M** is an *n-by-n* tridiagonal matrix. The point to be noted is that k_i is a medium property, and its value depends on the layer where i^{th} node is located if it is located on interface its value is taken as weighted average of k of surrounding layers. The size of matrix **M** depends on number of nodes taken. The situation where analytical solution is not available or cumbersome (e.g. for more than two layered earth) then very fine gridding is used so that obtained field vector u is very close to analytical result.

Solving PDE using Exponential Finite Difference Method (EFDM)

The CFDM formula derived in previous section holds good only when $u(x)$ can be approximated by a low degree polynomial. The situation where $u(x)$ is a weighted sum of exponential or trigonometric or hyperbolic functions, it will give approximate results only when nodes are very closely spaced. Exponential fitting approach deals with such a class of functions which have oscillatory or hyperbolic behavior.

The central difference formula for second derivative for non-uniform gridding can be written as,

$$u''(x_i) \approx \frac{1}{H_i^2}[A.u(x_{i-1}) + B.u(x_i) + C.u(x_{i+1})] \qquad (5)$$

To evaluate the coefficients **a**= [A, B, C] an operator $\mathcal{L}[h_i, h_{i+1}, a]$ is defined as follows,

$$\mathcal{L}[h_i, h_{i+1}, a]u(x_i) = u''(x_i) - \frac{1}{H^2{}_i}[Au(x_i - h_i) + Bu(x_i) + Cu(x_i + h_{i+1})] \qquad (6)$$

In case of EFDM, basis functions are $\{x^i.exp(\mu x)$, where i=0,1,2…..} and $u(x)$ is a linear superposition of these basis functions and we seek to find out such a set **a**=[A, B, C] so that $\mathcal{L}[h_i, h_{i+1}, a]u(x_i) = 0$. This implies that expression in equation 5 is appropriate. Therefore, this operator $\mathcal{L}$ is applied on different $u(x_i) = x^i\exp(\mu x)$, where i=0,1,2….., after solving these equations we get following set of coefficients **a**=[A, B, C],

$$C = \frac{Z}{\left[\{\eta_{-1}(Z_{i+1})-1\}+\{\eta_{-1}(Z_i)-1\}.\frac{\eta_0(Z_{i+1})}{\eta_0(Z_i)}.\sqrt{\frac{Z_{i+1}}{Z_i}}\right]},$$

$$A = \frac{\eta_0(Z_{i+1})}{\eta_0(Z_i)}.\sqrt{\frac{Z_{i+1}}{Z_i}}.C,$$

$$B = -\left[1+\frac{\eta_0(Z_{i+1})}{\eta_0(Z_i)}.\sqrt{\frac{Z_{i+1}}{Z_i}}\right].C \quad , \qquad (7)$$

where,

$$Z_i = \mu^2 h_i^2, \qquad Z = \mu^2 H^2, \qquad H^2 = \frac{1}{2}(h_i.h_{i+1})$$

$$\eta_0(Z_i) = \frac{1}{2\sqrt{Z_i}}\left[\exp\left(Z_i^{\frac{1}{2}}\right) - \exp\left(-Z_i^{\frac{1}{2}}\right)\right] \qquad \text{and}$$

$$\eta_{-1}(Z_i) = \frac{1}{2}\left[\exp\left(Z_i^{\frac{1}{2}}\right) + \exp\left(-Z_i^{\frac{1}{2}}\right)\right].$$

The above equation 7 gives the coefficient for EFDM with non-uniform grid system. These coefficients are required to be placed in the 1D Helmholtz equation to obtain fields $u(x_i)$ using EFDM,

Using equation 1 and equation 5

$$\frac{1}{H_i{}^2}[A.u(x_{i-1}) + B.u(x_i) + C.u(x_{i+1})] + k_i^2 u(x_i) = 0$$

$$A.u(x_{i-1}) + \{B + k_i^2 H_i{}^2\}.u(x_i) + C.u(x_{i+1}) = 0 \qquad (8)$$

This results in following matrix equation similar to the case of CFDM,

$$\mathbf{M_{ex}}.\mathbf{u_{ex}} = \mathbf{b} \qquad (9)$$

The structure of this matrix is also tridigoanal and bands will contain coefficients a = [A, B, C] corresponding to different nodes.

The derivation of set of coefficients in equation 7 is an important development which extends the existing results of Exponential Finite Difference approximation for uniform gridding given in Ixaru and Berghe, 2004. The basic aim of using non uniform gridding is that solving Helmholtz equation for layered earth medium demands grids to be finely spaced near the interfaces compared to that in the middle of layer.

Estimation of optimum value of parameter μ

A significant point of discussion in EFDM is the factor μ which is hidden in Z = $\mu^2 H^2$. The coefficients **a** = [A, B, C] as in equation 7 depend on Z, hence these, in turn, depend on μ. Let the electromagnetic field be assumed to be a linear combination of basis functions {1, *exp(±μx)*, *x.exp(±μx)*…}. Now, if field has some other frequency μ_1 (μ_1 μ) then coefficients **a** = [A, B, C] will not give the true response. So, the choice of μ is a crucial task in EFDM. If we have prior knowledge about field then μ can be chosen accordingly. It has been shown in results section that for homogeneous half space EFDM is far better than CFDM because μ is taken in accordance with the field. But for layered earth it is a critical task to obtain optimum μ value.

To obtain suitable μ, various experiments have been performed. For a given model, when μ is taken greater than 0.01 then solution diverges and if μ is taken to be less than 0.01 then solution converges, but value of L_1 norm of relative error $\xi_{exp} = \sum_n \frac{(u_{\exp} - u_{anl})}{u_{anl}}$ (where u_{exp} is computed field with EFDM and u_{anl} is analytical field) is large and it has minimum value at particular μ called as optimum μ.

After conducting experiments on random layered earth models, we defined the estimator for μ through model parameters itself as well as eigenvalue analysis of coefficient matrix which is as follows,

A. Using model parameters

An estimator of μ through model parameters is defined in terms of *pk*(*g*) (Ray, 2011),

$$pk(g) = \frac{\left(\sum_{j=1}^{n}(-1+i)*\frac{real\{k(j)\}^g}{d(j)^g}\right)}{\sum_{j=1}^{n}\frac{1}{d(j)^g}}, \quad for\ g = 1, 2, 3.... \qquad (10)$$

Here j runs from 1st layer to nth layer, *k* is a term that appears in 1D Helmholtz equation, *d*(*j*) is thickness of jth layer and *d*(*n*) = *total depth* $\sum_{j=1}^{n-1} d(j)$. It can be noted that the ratio *pk*(*g*) is some sort of weighted average of *k*. And μ is given as μ = imaginary part of *pk*(*g*) or negative of real part of *pk*(*g*).

Now we solve 1D Helmholtz equation for different layered earth models using EFDM for the values of μ lying between 10^{-3} to 10^{-13}. Here we show in figure 2 only two models M1 and M2 (given in results section) and corresponding computed L1 norm of relative error for CFDM and EFDM.

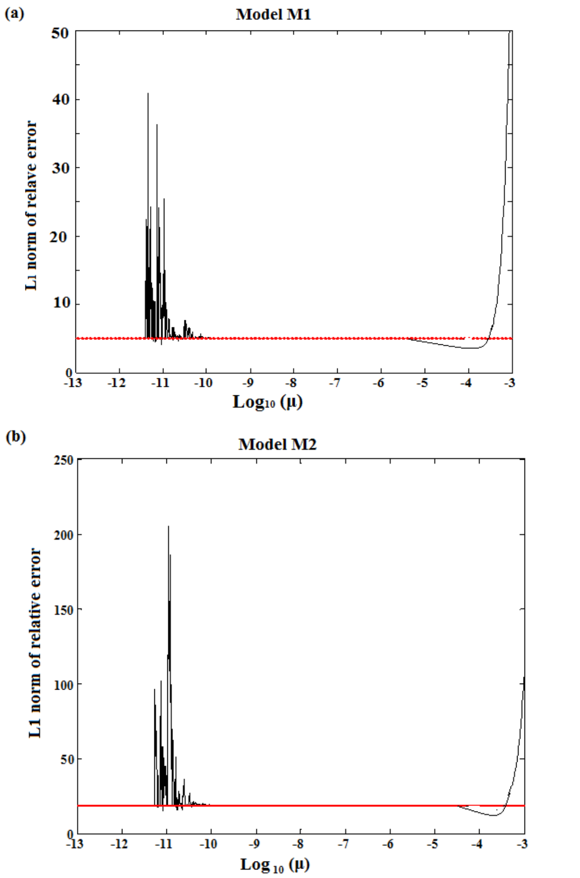

Figure 2: Variation of L1 norm of relative error ( ) with μ for (a) model M1 and (b) model M2. Red line shows error in CFDM and black line shows error in EFDM.

From figure 2 we observe that for the values of μ between 10^{-3} to 10^{-9}, the error norm is stable and has minimum value at particular μ within this range and between 10^{-10} and 10^{-12} error norm is unstable and highly oscillating. Using equation 10 the computed value of $pk(g)$ for model M1 and model M2 are as follows,

Model M_1:

$$pk(1) = -1.02(-1+i)\times 10^{-4},$$
$$pk(2) = -6.93(-1+i)\times 10^{-9},$$
$$pk(3) = -4.70(-1+i)\times 10^{-13}$$

Model M_2:

$$pk(1) = -1.59(-1+i)\times 10^{-4},$$
$$pk(2) = -1.68(-1+i)\times 10^{-8}$$
$$pk(3) = -1.78(-1+i)\times 10^{-12}$$

Therefore, from this estimator we found the range of μ between imaginary parts of $pk(3)$ x 10^3 to $pk(3)$ give unstable and highly oscillating result. But the range of μ between imaginary parts of $pk(1)$ x 10^{-1} to imaginary part of $pk(2)$ give stable result and has minimum error at optimum μ. But using this estimator we cannot find the optimum value of μ at which error norm is optimal. It is also observed that optimum value of μ depends on node spacing. Therefore, to find optimum value of μ we performed eigenvalue analysis of coefficient matrix.

B. Using Eigenvalue analysis of coefficient Matrix

The task of obtaining optimum value of μ through eigenvalue analysis is similar to Young's work of obtaining optimum value of ω to ensure fast convergence of Overrelaxed Gauss Seidel's method to solve linear system of equations (Young, 1950). Using same approach, we exhaustively analyzed numerous combination of eigenvalues of coefficient matrix for several classes of layered earth models to find optimum value of μ and we have identified following combination of largest and smallest eigenvalue of coefficient matrix gives optimum μ for all model,

$$\mu = \pm\frac{5}{3\sqrt{2}}.\frac{|\lambda_n|\times|S|^{3/2}}{|\lambda_1|.H} \qquad (11)$$

Where | | denotes absolute values, λ_1 is largest eigenvalue, λ_n is smallest eigenvalue of coefficient matrix, S is trace of coefficient matrix and $H^2 = \frac{1}{2}(h_{i+1}.h_i)$. This shows that value of μ depends on node spacing.

Using this definition of μ, we get optimum value of L_1 norm of error (ξ_{exp}) in EFDM for all models.

Results

To compare the two numerical schemes CFDM and EFDM, first both methods are applied on homogeneous half space for different nodes spacing then experiments are carried on layered earth models.

Response for uniform half space

Consider a homogeneous half space of thickness 34 km having resistivity of ρ=100 Ω-m and time period of signal is T=1 second. Boundary condition at the surface is taken to be u_0= 100(1+i) NC^{-1}.

Since for uniform half space we can solve 1D Helmholtz equation analytically and field is given as, $u_{ani}(x) = u_0\exp(ikx)$. 1D Helmholtz equation is also solved using CFDM and proposed EFDM. In EFDM, μ= *ik* is taken. After computing the field with both methods, we compute the L_1 norm of relative error i.e., $\xi = \sum_n \frac{|u_{cal} - u_{anl}|)}{u_{anl}}$, (where u_{cal} is computed field with either CFDM or EFDM) for different node spacing which is shown in figure 3.

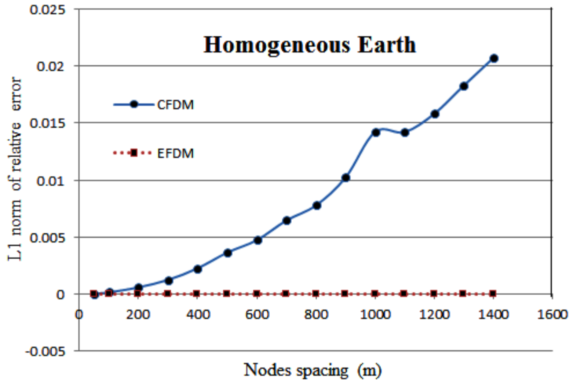

Figure 3: Variation of ξ (L_1 norm of relative error) with node spacing for homogeneous half space. Blue line shows error in CFDM and red line shows error in EFDM.

From figure 3 we observed that L_1 norm of relative error in EFDM is far less than CFDM and it nearly equals zero. Therefore, for homogeneous half space, EFDM gives nearly exact analytical solution. This is because the value of μ is taken to be same as frequency of field.

Response for layered Earth Models

The motivation to compute electromagnetic response of layered earth model through EFDM came from result of uniform half space. Layered earth is more realistic than uniform half space model of earth. The solution of 1D Helmholtz equation for layered earth obtained using both CFDM and EFDM with different node spacings. Since analytical solution is difficult to obtain for more than two layers, therefore, very fine gridding (nodes spacing of 25 m) is taken and response is computed using CFDM which can be considered as analytical solution.

Let at the surface, field value is taken as u_0=100(1+i) NC^{-1} and at the end point, field is considered to be very low ($u_{n+1} \approx 0.000001$). So the assumed boundary conditions constrained the resistivity of bottom layers to be very low compared to top three layers. Now proposed methods are tested on various models from which some of models along with their L1 norm of relative errors are as follows,

Model M_1: Five layered earth model with resistivities ρ1 = 856 Ω-m, ρ2 = 256 Ω-m, ρ3 = 658 Ω-m, ρ4 = 35 Ω-m, and ρ5= 22 Ω-m and thicknesses d_1 = 1.5 km, d_2 = 2.3 km, d_3 = 3.4 km, d_4 = 5.6 km, and d_5 = 20 km and total depth is 32.8 km.

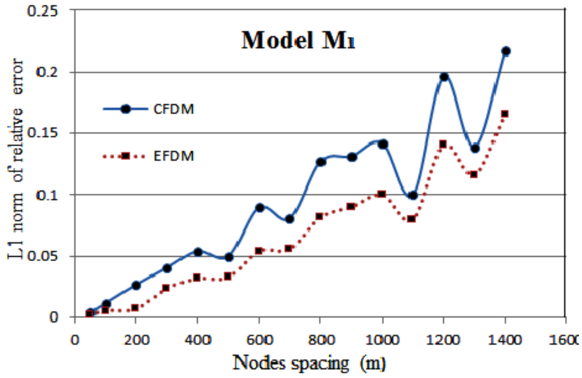

Figure 4: Variation of ξ (L1 norm of relative error) with node spacing for model M_1.

Here, we clearly see that relative error in EFDM is less than error in CFDM hence EFDM gives more accurate result as compared to CFDM. Here it should be noted that EFDM with 500 m nodes spacing gives results of same accuracy as we get from CFDM with 200 m nodes spacing. Similarly, EFDM with 700 m nodes spacing gives results of same accuracy as we get from CFDM with 400 m nodes spacing. EFDM with 900 m nodes spacing gives same result as CFDM with 600 m nodes spacing and so on.

Model M_2: Five layered earth model with resistivities ρ_1 = 353 Ω-m, ρ_2 = 16 Ω-m, ρ_3 = 138 Ω-m, ρ_4 = 5 Ω-m, and ρ_5 = 10 Ω-m and thicknesses d_1 = 1.7 km, d_2 = 2.1 km, d_3 = 2.9 km, d_4 = 2.5 km, and d_5 = 20 km and total depth is 29.2 km.

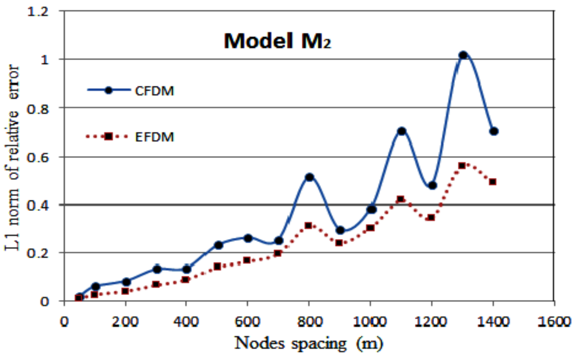

Figure 5: Variation of ξ (L1 norm of relative error) with node spacing for model M_2.

In this EFDM with 500 m nodes spacing gives results of same accuracy as we get from CFDM with 300 m nodes spacing. Similarly, EFDM with 900 m nodes spacing gives results of same accuracy as we get from CFDM with 600 m nodes spacing. EFDM with 1400 m nodes spacing gives results of same accuracy as we get from CFDM with 800 m nodes spacing.

Model M_3: Five layered earth model with resistivities ρ_1 = 785 Ω-m, ρ_2 = 125 Ω-m, ρ_3 = 367 Ω-m, ρ_4 = 23 Ω-m, and ρ_5 = 12 Ω-m and thicknesses d_1 = 0.9 km, d_2 = 1.8 km, d_3 = 2.3 km, d_4 = 1.9 km, and d_5 = 20 km and total depth is 26.9 km.

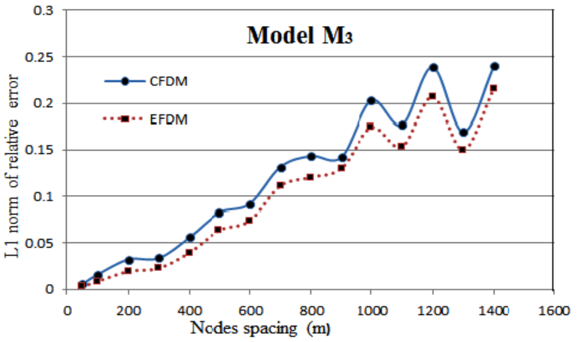

Figure 6: Variation of ξ (L1 norm of relative error) with node spacing for model M_3.

In this EFDM with 900 m nodes spacing gives results of same accuracy as we get using CFDM with 700 m nodes spacing. EFDM with 1200 m nodes spacing gives results of same accuracy as we get using CFDM with 1000 m nodes spacing.

Model M_4: Four layered earth model with resistivities ρ_1 = 489 Ωm, ρ_2 = 445 Ωm, ρ_3 = 646 Ωm, and ρ_4 = 10 Ωm and thicknesses d_1 = 1.5 km, d_2 = 2.3 km, d_3 = 3.4 km and d_4 = 20 km and total depth is 27.2 km.

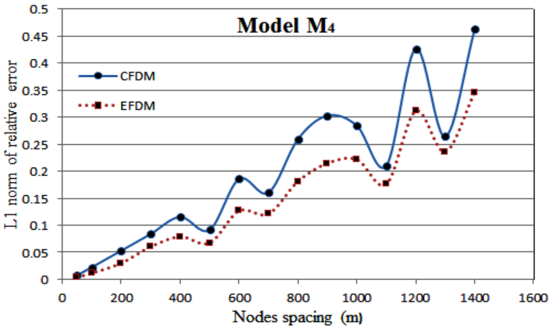

Figure 7: Variation of ξ (L_1 norm of relative error) with node spacing for model M_4.

In this EFDM with 500 m nodes spacing gives results of same accuracy as we get using CFDM with 300 m nodes spacing. Similarly, EFDM with 800 m nodes spacing gives results of same accuracy as we get using CFDM with 600 m nodes spacing. EFDM with 1200 m nodes spacing gives results of same accuracy as we get from CFDM with 900 m nodes spacing.

From these error plots of models M_1 to M_4 we can easily see that error in EFDM method is less as compared to CFDM hence EFDM gives more accurate result as compared to CFDM and we can use coarse grid for EFDM to obtain same accuracy of result as we get using CFDM. As a result EFDM reduces the time and cost of computation. Here it should be noted that norm of relative error has oscillating character at coarse nodes spacing. This is arises due to numerical noise because nodes spacing is coarse enough in respect to thickness of layers.

Conclusions

The Exponential Finite Difference Method (EFDM) is developed using exponential basis function. The crucial part in EFDM is the selection of optimum value of μ. Using eigenvalue analysis of coefficient matrix we defined formula for optimum μ that gives best result for all models. Both the numerical scheme Classical Finite Difference Method (CFDM) and EFDM are applied on various layered earth models for solving 1D Helmholtz equation. From the experiments we observed that EFDM gives better results than CFDM. For uniform half space, EFDM gives result nearly similar to analytical results and far better than CFDM. For layered earth EFDM gives more accurate result in comparison to CFDM. This implies that we can use coarse grid for EFDM that will give same accuracy of results as we get using CFDM with fine grid. Therefore, using EFDM we can sufficiently reduce time and cost of computation. When we deal with 2D/3D problems, the superiority of proposed EFDM would be much greater than CFDM in respect of time and cost of computations.

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