9th Biennial International Conference
& Exposition on Petroleum Geophysics

"HYDERABAD 2012"

P-261

Poroelastic Seismic Boundary Conditions

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Summary

In the context of seismic wave propagation, the boundary conditions for two porous media in contact are not yet well established. There are two competing schools of thought, namely, due to Deresiewicz and Skalak (1963) and de la Cruz and Spanos (1989). In here, a critical assessments of the both are made. In this process a novel result is established that at a boundary total, solid, and fluid tractions are continuous as well as equal to each other.

Introduction

The boundary conditions define the field quantities across the surface of a material discontinuity. For the case wherein there is no imbedded source or sink of energy at the interface, they are simply the statements of the conservation of the fundamental quantities, namely, mass, linear momentum, angular momentum and energy.

In classical elasticity, for the case of a welded contact, boundary conditions are stated as the continuity of the velocity field and the traction field. The conservation of mass across an inter- face requires the continuity of normal component of velocity. The continuity of tractions is the statement about the conservation of linear momentum across an interface. The conservation of angular momentum further requires the continuity of tangential components of velocity field; although in literature this continuity is often ascribed to kinematic requirement of "no-slip". The continuity of energy flux is required for conservation of energy, which is automatically satisfied on account of the continuity of velocity and traction fields.

For a porous-porous welded contact, the two interacting con- and "partially open pore". The latter contains an adjustable parameter that is known as "interface permeability".

The other competing set of boundary conditions is due to de la Cruz and Spanos (1989). They have suggested that macroscopic interface has to be the surface across with total mass is conserved. They show that across this interface, the conservation of mass and linear momentum yield the continuity of nor- mal component of mass weighted vector sum of the solid and fluid velocities and total tractions; for the continuity of the tangential components of that mass weighted velocity field they invoke "no slip" condition. By utilizing Newton's third law of motion, they have quantified how the stresses on each phase interact with the stress on each of the phases across the interface and have developed two additional conditions on tractions. These equations contain a parameter to describe "over- lap" among the phases on two sides of an interface.

A critical overview of the Deresiewick and Skalak (1963) boundary conditions, along with its extension that includes fluid viscous stress tensor part, is presented in the next section. There- after, the de la Cruz and Spanos (1989) boundary conditions along with an extension, are described. This extension recast them in a form in which the "overlap" parameter is no longer present. In this form it simply states that across the boundary there is equalization of all forces along with continuity of the velocity field associated with the total mass flux.

List of Field Quantites and Parameters

η		porosity
ρ_s		solid density
ρ_f		fluid density
ρ	$= (1-\eta)\rho_f + \eta\rho_f$	total density
m_s	$= \frac{(1-\eta)\rho_s}{\rho}$	solid mass fraction
m_f	$= \frac{\eta\rho_f}{\rho}$	fluid mass fraction
$\dot{u}^s_j$		solid velocity field
$\dot{u}^f_j$		fluid velocity field
$\dot{u}^f_\perp$		fluid velocity normal to the interface
$\dot{w}_j$		fluid filtration velocity field
$\dot{w}_\perp$		fluid filtration velocity normal to the interface
$\dot{u}_j$	$= m_s\dot{u}^s_j + m_f\dot{u}^f_j$	velocity of the total mass flux
τ^s_{jk}		solid stress tensor
τ^f_{jk}		fluid stress tensor
p^f	$= -\frac{\tau^f_{jj}}{\eta}$	fluid pressure
τ_{jk}	$= \tau^s_{jk} + \tau^f_{jk}$	total stress tensor
σ^s_{jk}	$= \frac{\tau^s_{jk}}{1-\eta}$	solid stress tensor (per unit area)
σ^f_{jk}	$= \frac{\tau^f_{jk}}{\eta}$	fluid stress tensor (per unit area)

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DERESIEWICZ AND SKALAK (1963) BOUNDARY CONDITIONS

The boundary conditions due to Deresiewicz and Skalak (1963) are based upon of the conservation of total energy. They showed that to conserve total energy across an interface the normal energy flux has to remain continuous, and for which they expressed as

$$\left\langle \left(\dot{u}^{s}_{j} \tau^{s}_{jk} + \dot{u}^{f}_{j} \tau^{f}_{jk} \right) \hat{n}_k \right\rangle = 0 \tag{1}$$

must hold true. Here, $\hat{n}_k$ denotes the unit normal to the interface separating two dissimilar porous media and the bra-ket symbol, $\langle\ \rangle$, represents the jump in the quantity within its argument. To have consistency with the classical Biot theory, the viscous shear stress part of the fluid stress tensor was not taken into consideration by them. Retaining only the hydrostatic part of the fluid stress tensor, they took

$$\tau^{f}_{jk} = -\eta \mathrm{p}^{f} \delta_{jk} \tag{2}$$

where η is the porosity and p^f is the fluid pressure. At first, they assumed (implicitly) that the two terms in equation (**??**) have to be individually continuous, i. e.,

$$\left\langle \dot{u}^{s}_{j} \tau^{s}_{jk} \hat{n}_k \right\rangle = 0, \tag{3}$$

and

$$\left\langle \delta_{jk} \dot{u}^{f}_{j} \hat{n}_k \eta \mathrm{p}^{f} \right\rangle = 0. \tag{4}$$

$\dot{u}^s_j$ and $\tau^s_{jk}\hat{n}_k$ are the solid phase velocity field and the traction vector, respectively. The term $\delta_{jk}\dot{u}^f_j\hat{n}_k = \dot{u}^f_k\hat{n}_k \equiv \dot{u}^f_\perp$ is the component of fluid velocity normal to the interface. Clearly, the expression in equation (**??**) is the dot product of the solid velocity field vector and the solid phase traction vector. Thus, equation (**??**) is stating the continuity of energy flux of solid phase. Likewise, the expression in equation (**??**) is the dot product of the normal component of fluid velocity field vector and the porosity weighted fluid pressure, so equation (**??**) is the statement of the continuity of energy flux of the fluid phase. For equations (**??**) and (**??**) to hold true, Deresiewicz and Skalak (1963) proposed that each element of the dot product therein must be continuous. Those amount to the continuity of solid velocity field, solid traction vector, normal component of fluid velocity and porosity weighted fluid pressure,

$$\left\langle \dot{u}^{s}_{j} \hat{n}_k \right\rangle = 0, \tag{5}$$

$$\left\langle \tau^{s}_{jk} \hat{n}_k \right\rangle = 0, \tag{6}$$

$$\left\langle \dot{u}^{f}_{\perp} \right\rangle = 0, \tag{7}$$

$$\left\langle \eta \mathrm{p}^{f} \right\rangle = 0. \tag{8}$$

However, they concluded that the above set of boundary conditions are of limited scope. They asserted that the relative normal flow of fluid with respect to solid frame, $\eta\left(\dot{u}^f_\perp - \dot{u}^s_\perp\right)$, must remain continuous across the interface to conserve mass of fluid across the interface. This constrain is compatible with equations (**??**) through (**??**) only if porosity remains unchanged across the interface or there is no relative motion between fluid and solid-frame at the interface. The quantity

$$\eta \left(\dot{u}^{f}_{j} - \dot{u}^{s}_{j} \right) \equiv \dot{w}_j \tag{9}$$

is recognized as "fluid filtration velocity". The choice of continuity of the normal component of fluid filtration velocity suggests that solid-solid contact surface has been taken as the interface. The cartoon illustrations for porous interface presented in Deresiewicz and Skalak (1963) are also suggestive of that.

(a) Open pore case

In order to incorporate "fluid filtration velocity" into the statement of energy conservation, they re-wrote the expression of energy flux. In equation (**??**) the addition and subtraction of the term $\dot{u}^s_j \eta \mathrm{p}^f \delta_{jk}$ yields

$$\left\langle \left(\dot{u}^{s}_{j} \tau^{s}_{jk} - \dot{u}^{s}_{j} \eta \mathrm{p}^{f} \delta_{jk} + \dot{u}^{s}_{j} \eta \mathrm{p}^{f} \delta_{jk} - \dot{u}^{f}_{j} \eta \mathrm{p}^{f} \delta_{jk} \right) \hat{n}_k \right\rangle. \tag{10}$$

By regrouping terms for total stress, $\tau_{jk} = \tau^s_{jk} - \eta \mathrm{p}^f \delta_{jk}$, and filtration velocity (see equation **??**) it is

$$\left\langle \left(\dot{u}^{s}_{j} \underline{\left(\tau^{s}_{jk} - \eta \mathrm{p}^{f} \delta_{jk} \right)} - \underline{\eta \left(\dot{u}^{f}_{j} - \dot{u}^{s}_{j} \right)} \mathrm{p}^{f} \delta_{jk} \right) \hat{n}_k \right\rangle, \tag{11}$$

or

$$\left\langle \left(\dot{u}^{s}_{j} \tau_{jk} - \dot{w}_j \mathrm{p}^{f} \delta_{jk} \right) \hat{n}_k \right\rangle. \tag{12}$$

In the above, assuming the continuity of each term individually they proposed

$$\left\langle \dot{u}^{s}_{j} \right\rangle = 0, \tag{13}$$

$$\left\langle \tau_{jk} \hat{n}_k \right\rangle = 0, \tag{14}$$

$$\left\langle \dot{w}_\perp \right\rangle = 0, \tag{15}$$

$$\left\langle \mathrm{p}^{f} \right\rangle = 0. \tag{16}$$

They asserted that these are the boundary conditions for the case when fluid is able to freely flow across the interface and coined the named "open pore" to this set.

It should be noted that from equation (**??**), which is a one scalar equation, the deduction 8 equations, (**??**) through (**??**), is heuristic. It presupposes that the dot product of solid velocity field vector and total traction vector, $\dot{u}^s_j \tau_{jk} \hat{n}_k$, and product of normal component of filtration velocity and fluid pressure, $\delta_{jk}\dot{w}_j\hat{n}_k\mathrm{p}^f$, are continuous. If these quantities are conserved across the interface, they should represent some conservative fluxes. However, it is not apparent what physical meaning can be ascribed to these fluxes.

(b) Partially open pore case

Deresiewicz and Skalak (1963) further suggested that there shall be cases when fluid is not completely free to flow across the interface.

They argued that when the hydraulic contact between two porous media is imperfect, the boundary condition (**??**) will not hold true because of the difference in fluid pressure across the interface.They generalized equation (**??**) by proposing that the differential fluid pressure must be linearly related to the normal component of filtration velocity at the interface, i. e.,

$$\left\langle \mathrm{p}^{f} \right\rangle = \frac{1}{k_{\mathrm{DS}}} \dot{w}_\perp \tag{17}$$

k_{DS} is the proportionality constant which has the dimensions of permeability and it is known as "interface permeability". It is taken to span from $0 \leq k_{\mathrm{DS}} \leq \infty$. $k_{\mathrm{DS}} = \infty$ represents the open pore case as it yields equation (**??**). The equations (**??**) through (**??**) and (**??**) are the final set of boundary conditions proposed by Deresiewicz and Skalak.

However, according to them, $k_{\mathrm{DS}} = 0$ corresponds to the seal pore case. For that they suggested to take equations (**??**) and (**??**) as is, and to replace equations (**??**) and (**??**) by the conditions of the vanishing normal component of fluid filtration velocity at the interface for each side.

(c) Generalization for fluid viscous stress tensor
In order to be consistent with the Navier Strokes equation, the Biot theory (Biot, 1956) must include fluid viscous stress term into its constitutive equations (Sahay 2008; G. Mavko et al., 2009). For consistency with this viscosity-corrected framework of the Biot theory, the viscous shear stress part of the fluid stress tensor, which was dropped in equation (**??**), has to be re-incorporated and the boundary conditions have to be reworked. Accordingly, one finds that in the boundary equations (**??** through **??** and **??**) the fluid pressure term has to be replaced by the fluid stress tensor term and the normal component of fluid filtration velocity has to be replaced by the filtration velocity vector. The derivations are straightforward. The viscosity generalized Deresiewicz and Skalak (1963) boundary conditions read

$$\left\langle \dot{\mathrm{u}}^{\mathrm{s}}_{j} \right\rangle = 0, \qquad (18)$$

$$\left\langle \tau_{jk}\hat{n}_{k} \right\rangle = 0, \qquad (19)$$

$$\left\langle \dot{\mathrm{w}}_{j} \right\rangle = 0, \qquad (20)$$

$$\left\langle \sigma^{\mathrm{f}}_{\mathrm{jk}}\hat{\mathrm{n}}_{\mathrm{k}} \right\rangle = \frac{1}{k_{\mathrm{DS}}}\dot{\mathrm{w}}_{j}, \qquad (21)$$

where $\sigma^{\mathrm{f}}_{jk} = \frac{\tau^{\mathrm{f}}_{jk}}{\eta}$, and it is the fluid stress per unit area.

DE LA CRUZ AND SPANOS (1989) BOUNDARY CONDITIONS

The boundary conditions due de la Cruz and Spanos (1989) are based upon conservation of total mass and total linear momentum and the application of Newton's third law of motion to describe how the stresses on each phase interact with the stress on each of the phases across the boundary.

(a) Continuity of total mass
de la Cruz and Spanos started with the well established equations of mass balance for multi-phasic medium. For the solid and fluid bi-phasic medium, those equations read as below

$$\frac{\partial}{\partial t}((1-\eta)\rho_{\mathrm{s}}) = \partial_{j}((1-\eta)\rho_{\mathrm{s}}\dot{\mathrm{u}}^{\mathrm{s}}_{j}), \qquad (22)$$

$$\frac{\partial}{\partial t}(\eta\rho_{\mathrm{f}}) = \partial_{j}(\eta\rho_{\mathrm{f}}\dot{\mathrm{u}}^{\mathrm{f}}_{j}). \qquad (23)$$

They showed that by adding the above two equations one obtains

$$\frac{\partial}{\partial t}(\underbrace{(1-\eta)\rho_{\mathrm{s}}+\eta\rho_{\mathrm{f}}}_{\rho}) = \partial_{j}(\underbrace{(1-\eta)\rho_{s}\dot{\mathrm{u}}^{\mathrm{s}}_{j}+\eta\rho_{f}\dot{\mathrm{u}}^{\mathrm{f}}_{j}}_{\rho\ (m_{\mathrm{s}}\dot{\mathrm{u}}^{\mathrm{s}}_{j}+m_{\mathrm{f}}\dot{\mathrm{u}}^{\mathrm{f}}_{j})}). \qquad (24)$$

The term, $(1-\eta)\rho_{\mathrm{s}}+\eta\rho_{\mathrm{f}}$, on the left-hand side of equation, is recognized as total density of poro-continuum, ρ. On right-hand side of equation, ρ is factored out and the remaining term, with the aid of solid ($m_{\mathrm{s}} = \frac{(1-\eta)\rho_{\mathrm{s}}}{\rho}$) and fluid ($m_{\mathrm{f}} = \frac{\eta\rho_{\mathrm{f}}}{\rho}$) mass fractions, is written as $m_{\mathrm{s}}\dot{\mathrm{u}}^{\mathrm{s}}_{j}+m_{\mathrm{f}}\dot{\mathrm{u}}^{\mathrm{f}}_{j}$, which is the vector sum of mass weighted solid and fluid velocity fields. They pointed out that since the above equation is obviously the equation of balance of total mass of poro-continuum, the vector sum of mass weighted solid and fluid velocity fields has to be recognized as the velocity associated with the linear momentum flux. Thereupon, they suggested that if total mass of poro-continuum has to conserve across the interface, then the normal component of this velocity field must remain continuous, i. e.,

$$\left\langle m_{\mathrm{s}}\dot{\mathrm{u}}^{\mathrm{s}}_{\perp}+m_{\mathrm{f}}\dot{\mathrm{u}}^{\mathrm{f}}_{\perp} \right\rangle = 0. \qquad (25)$$

They argued that the macroscopic interface between two porous media has to be taken as the surface across which total mass is conserved, not the fluid-fluid or solid-solid contact surfaces. Furthermore, on the basis of the macroscopic nonslip, they assumed that tangential component of this velocity field should also remain continuous which lead to

$$\left\langle m_{\mathrm{s}}\dot{\mathrm{u}}^{\mathrm{s}}_{j}+m_{\mathrm{f}}\dot{\mathrm{u}}^{\mathrm{f}}_{j} \right\rangle = 0. \qquad (26)$$

(b) Continuity of total linear momentum
de la Cruz and Spanos (1989) showed that by adding the equations of motion for solid-frame and fluid constituent one obtains

$$\frac{\partial}{\partial t}(\rho(m_{\mathrm{s}}\dot{\mathrm{u}}^{\mathrm{s}}_{j}+m_{\mathrm{f}}\dot{\mathrm{u}}^{\mathrm{f}}_{j})) = \partial_{k}(\tau^{\mathrm{s}}_{jk}+\tau^{\mathrm{f}}_{jk}). \qquad (27)$$

They argued since the left-hand side term is the rate of change of total linear momentum, equation (**??**) is the statement of conservation total of linear momentum, therefore, the sum, $\tau^{\mathrm{s}}_{jk}+\tau^{\mathrm{f}}_{jk}$, has to be taken as the total stress, τ_{jk}, of poro-continuum. Thereupon, they suggested that if the total linear momentum has to be conserved across the interface, then total traction of poro contimuum must remain continuous, i. e.,

$$\left\langle \left(\tau^{\mathrm{s}}_{jk}+\tau^{\mathrm{f}}_{jk}\right)\hat{n}_{k} \right\rangle = 0. \qquad (28)$$

(c) Newton's third law of motion and balance of phasic forces
Equation (**??**) implies that the total forced exerted on the interface by a side is balanced by an equal and opposite force exerted on the interface by the other side. This is the statement of Newton's third law of motion. de la Cruz and Spanos (1989) have asserted that as Newton's third law is operational on the totality of the phasic forces, it must hold true for how phasic forces are individually balanced at the interface. On this basis they suggested two additional conditions on tractions as below.

At interface, a given phase from one side overlaps with the both phases on the other side. Say, η_{a} be average area, per unit circle, of the fluid phase in the side "*a*" that is in contact with the fluid and solid phases in the side "*b*". Without loss of generality, it may be viewed as the unperturbed porosity of the side "*a*". Let P and Q be the average area of the overlap of the fluid in the side "*a*" with the fluid and solid phases in the side "*b*", respectively. Then, force exerted by medium "*b*" on the fluid phase in "*a*" is

$$P\sigma_{jk}^{\mathrm{f}(b)}\hat{n}_k + Q\sigma_{jk}^{\mathrm{s}(b)}\hat{n}_k, \tag{29}$$

where $\sigma_{jk}^{\mathrm{f}(b)}\hat{n}_k$ and $\sigma_{jk}^{\mathrm{s}(b)}\hat{n}_k$ stand for force exerted, per unit area, by fluid and solid phases on the side "*b*", respectively. Clearly, in the view of Newton's third law motion, this must be balanced by forces exerted by the fluid phase in the side "*a*", i. e.,

$$P\sigma_{jk}^{\mathrm{f}(b)}\hat{n}_k + Q\sigma_{jk}^{\mathrm{s}(b)}\hat{n}_k = \eta_a\sigma_{jk}^{\mathrm{f}(a)}\hat{n}_k, \tag{30}$$

where $\sigma_{jk}^{\mathrm{f}(a)}\hat{n}_k$ is the force exerted, per unit area, by the fluid phase on the side "*b*". The reciprocal interaction, the balance of forces exerted by medium "*a*" on fluid in medium "*b*" and vice-versa, results into

$$P\sigma_{jk}^{\mathrm{f}(a)}\hat{n}_k + R\sigma_{jk}^{\mathrm{s}(a)}\hat{n}_k = \eta_b\sigma_{jk}^{\mathrm{f}(b)}\hat{n}_k, \tag{31}$$

where η_b is the average area per unit circle of the fluid phase in the side "*b*" and R is the average area of overlap of the fluid in the side "*b*" with the solid phase in the side "*a*".

By definition, $P+Q=\eta_a$ and $P+R=\eta_b$. Setting the average area of overlap of the fluids on two side, $P=\eta_a\eta_b\beta$, where β spans from $0\leq\beta\leq\frac{1}{\eta_{[a/b]}}$ and $\eta_{[a/b]}$ stands for the greater of η_a and η_b, one finds $Q=\eta_a(1-\eta_b\beta)$ and $R=\eta_b(1-\eta_a\beta)$. Here, β is viewed as the measure of overlap or alignment. Using these definitions of overlap areas, and employing the notations of phasic stresses

$$\begin{array}{ll} \tau_{jk}^{\mathrm{f}(a)} = \eta_a\sigma_{jk}^{\mathrm{f}(a)}, & \tau_{jk}^{\mathrm{s}(a)} = (1-\eta_a)\sigma_{jk}^{\mathrm{s}(a)}, \\ \tau_{jk}^{\mathrm{f}(b)} = \eta_b\sigma_{jk}^{\mathrm{f}(b)}, \quad \text{and} & \tau_{jk}^{\mathrm{s}(b)} = (1-\eta_b)\sigma_{jk}^{\mathrm{s}(b)}, \end{array} \tag{32}$$

equations (**??**) and (**??**) are written as

$$\tau_{jk}^{\mathrm{f}(a)}\hat{n}_k = \eta_a\beta\tau_{jk}^{\mathrm{f}(b)}\hat{n}_k + \frac{\eta_a(1-\eta_b\beta)}{1-\eta_b}\tau_{jk}^{\mathrm{s}(b)}\hat{n}_k, \tag{33}$$

$$\tau_{jk}^{\mathrm{f}(b)}\hat{n}_k = \eta_b\beta\tau_{jk}^{\mathrm{f}(a)}\hat{n}_k + \frac{\eta_b(1-\eta_a\beta)}{1-\eta_a}\tau_{jk}^{\mathrm{s}(a)}\hat{n}_k. \tag{34}$$

Likewise, the balance of forces exerted by the solid at one side upon the medium on the opposite side lead to

$$\tau_{jk}^{\mathrm{s}(a)}\hat{n}_k = (1-\eta_a\beta)\tau_{jk}^{\mathrm{f}(b)}\hat{n}_k + \left(1-\frac{\eta_a(1-\eta_b\beta)}{1-\eta_b}\right)\tau_{jk}^{\mathrm{s}(b)}\hat{n}_k, \tag{35}$$

$$\tau_{jk}^{\mathrm{s}(b)}\hat{n}_k = (1-\eta_b\beta)\tau_{jk}^{\mathrm{f}(a)}\hat{n}_k + \left(1-\frac{\eta_b(1-\eta_a\beta)}{1-\eta_a}\right)\tau_{jk}^{\mathrm{s}(a)}\hat{n}_k. \tag{36}$$

The addition of equation (**??**) with equation (**??**) and, as well as, the addition of equation (**??**) with equation (**??**) yields equation (**??**). Thus, the sets of equations (**??**-**??**) and (**??**-**??**) are derivable from each other. So these two sets are not independent of each other.

de la Cruz and Spanos (1989) have proposed that equations (**??**) and (**??**) together with either (**??**-**??**) or (**??**-**??**) are to be taken as the set of boundary conditions.

(d) The equalization of all forces: Continuity of total, fluid and solid tractions

Upon subjecting equations (**??**) and (**??**) to the continuity of total traction (**??**), one finds that they are simply the statement about continuity of fluid traction and solid traction acting on per unit area. The derivations are shown below.

By using the identities $\tau_{jk}^{\mathrm{f}(a)} + \tau_{jk}^{\mathrm{s}(a)} = \tau_{jk}^{(a)}$, and $\tau_{jk}^{\mathrm{f}(b)} + \tau_{jk}^{\mathrm{s}(b)} = \tau_{jk}^{(b)}$ in equations (**??**) and (**??**), the solid stress parts are eliminated. Upon further rearrangements they read, in terms of fluid tractions and total tractions, as

$$\tau_{jk}^{\mathrm{f}(a)}\hat{n}_k - \frac{\eta_a(\beta-1)}{1-\eta_b}\tau_{jk}^{\mathrm{f(b)}}\hat{n}_k = \eta_a\left(1-\frac{\eta_b(\beta-1)}{1-\eta_b}\right)\tau_{jk}^{\mathrm{(b)}}\hat{n}_k, \tag{37}$$

$$\tau_{jk}^{\mathrm{f}(b)}\hat{n}_k - \frac{\eta_b(\beta-1)}{1-\eta_a}\tau_{jk}^{\mathrm{f(a)}}\hat{n}_k = \eta_b\left(1-\frac{\eta_a(\beta-1)}{1-\eta_a}\right)\tau_{jk}^{\mathrm{(a)}}\hat{n}_k. \tag{38}$$

On the basis of the continuity of total traction (**??**), setting $\tau_{jk}^{(a)}\hat{n}_k = \tau_{jk}^{(b)}\hat{n}_k \equiv \tau_{jk}\hat{n}_k$ in the above and solving for the fluid tractions yields

$$\tau_{jk}^{\mathrm{f}(a)}\hat{n}_k = \eta_a\tau_{jk}\hat{n}_k, \tag{39}$$

$$\tau_{jk}^{\mathrm{f}(b)}\hat{n}_k = \eta_b\tau_{jk}\hat{n}_k. \tag{40}$$

Recalling that $\tau_{jk}^{\mathrm{f}}\hat{n}_k$ is the force extered by the fluid phase upon η part of the unit area (see equation **??**), the above pair of equations may be viewed as

$$\sigma_{jk}^{\mathrm{f}(a)}\hat{n}_k = \tau_{jk}\hat{n}_k, \tag{41}$$

$$\sigma_{jk}^{\mathrm{f}(b)}\hat{n}_k = \tau_{jk}\hat{n}_k. \tag{42}$$

Those, in turn, imply the continuity of $\sigma_{jk}^{\mathrm{f}}\hat{n}_k$, the force exerted by the fluid phase upon an unit area

$$<\sigma_{jk}^{\mathrm{f}}\hat{n}_k> = 0. \tag{43}$$

Also, by using the definitions $\tau_{jk}^{\mathrm{f}(a)} = \tau_{jk}^{(a)} - \tau_{jk}^{\mathrm{s}(a)}$ and $\tau_{jk}^{\mathrm{f}(b)} = \tau_{jk}^{(b)} - \tau_{jk}^{\mathrm{s}(b)}$, in conjunction with the identity that at interface $\tau_{jk}^{(a)}\hat{n}_k = \tau_{jk}^{(b)}\hat{n}_k \equiv \tau_{jk}\hat{n}_k$, (**??**) and (**??**) are

$$\tau_{jk}^{\mathrm{s}(a)}\hat{n}_k = (1-\eta_a)\tau_{jk}\hat{n}_k, \tag{44}$$

$$\tau_{jk}^{\mathrm{s}(b)}\hat{n}_k = (1-\eta_b)\tau_{jk}\hat{n}_k. \tag{45}$$

Realizing that that $\tau_{jk}^{\mathrm{s}}\hat{n}_k$ is the force extered by the solid phase upon $(1-\eta)$ part of the unit area (see equation **??**), the above pair of equations are

$$\sigma_{jk}^{\mathrm{s}(a)}\hat{n}_k = \tau_{jk}\hat{n}_k, \tag{46}$$

$$\sigma_{jk}^{\mathrm{s}(b)}\hat{n}_k = \tau_{jk}\hat{n}_k, \tag{47}$$

and those give the continuity of $\sigma_{jk}^{\mathrm{s}}\hat{n}_k$, the force exerted by the solid phase upon an unit area,

$$<\sigma_{jk}^{\mathrm{s}}\hat{n}_k> = 0. \tag{48}$$

Equations (**??**) and (**??**) the final form of equations (**??**) and (**??**). It is to be noted that the alignment parameter, β, is no longer present in this form.

Thus, Newton's third law of motion amounts to the continuity of the solid and the fluid phasic forces of the two sides at the interface as well as the equalization of the solid and fluid phasic forces on each side. That is, at the interface all forces are equal to one another, and that holds true irrespective of the alignment of the pores at the interface.

(e) The restatement of de la Cruz and Spanos (1989) boundary conditions

In the view of the equalization of all force at the interface, the continuity of the velocity field associate with total mass flux, equation (**??**), together with continuity of total, fluid and solid tractions, which are equations (**??**), (**??**) and (**??**) respectively, constitute the proper representation of the de la Cruz and Spanos (1989) boundary conditions.

CONCLUSIONS

Deresiewicz and Skalak (1963) boundary conditions are based upon conservation of total energy across the interface. This work preferentially treats the solid/solid contact surface as the interface and, on physical grounds, assumes the continuity of the normal component of so called "fluid filtration velocity" across it. Also, the two algebraic terms constituting the expression of the energy flux are assumed to be individually continuous across the interface, since their sum is supposed to be continuous. No physical meaning can be ascribed to the separate continuity of these two terms, although, mathematically such a continuity is a possibility. Under these assumptions, continuity conditions on solid velocity field, normal component of fluid filtration velocity, and total stress and jump condition on fluid pressure are established. Furthermore, the jump condition on fluid pressure introduces so-called "interface permeability" which supposedly spans from zero to infinity. It can be viewed only as an adjustable parameter, because it cannot be measured independently.

de la Cruz and Spanos (1989) framework regards a boundary between two porous media as the surface across which total mass of poro-continuum is conserved. In here, by invoking the conservation of total mass and total momentum, continuity of the velocity field (associated with the total mass flux) and the total stress are established. Furthermore, by using the concept of alignment, how the stresses on each phase interact with the stress on each of the phases across the interface is established. That results into two additional sets of conditions on tractions with a parameter which is a measure of the pore alignment.

It is shown here that de la Cruz and Spanos (1989) framework can be recast in a form in which the alignment parameter is no longer present. In fact, its three sets on boundary conditions involving tractions are simply the statement of the continuity of the solid and the fluid phasic forces across the interface as well as the equalization of the solid and fluid phasic forces on each side of interface. That is, at the interface all forces are equal to one another, and that holds true irrespective of the way pores are aligned at the interface.

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